Stability Analysis of Flexible Spacecraft via the Method of Integral Coordinates

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This paper presents an extension of the method of integral coordinates to the stability analysis of hybrid dynamical systems containing two- and three-dimensional elastic members. A common approach to such problems consists of system discretization by means of series representation in terms of admissible functions, an approach involving lengthy computations. By contrast, the method of integral coordinates yields stability criteria with relative ease and in a form that permits ready physical interpretation. As an application, the attitude stability of a force-free satellite containing a membrane is investigated and closed-form stability criteria are derived.

I. Introduction

THE motion of spinning satellites with distributed elastic parts can be described by a set of ordinary differential equations for the rotational motion of a given reference frame and a set of partial differential equations for the elastic motion relative to that frame. Such a system of differential equations has come to be known as a hybrid dynamical system.

The stability of hybrid systems has been the subject of numerous investigations. Meirovitch and Nelson[1] investigated the stability of a spinning satellite containing elastic rods along the spin axis by means of an infinitesimal analysis. The displacement of the elastic members is represented in Ref. 1 as a series of assumed modes multiplying time-dependent generalized coordinates. This is a discretization procedure leading to a system that consists entirely of ordinary differential equations with the order of the equations depending upon the number of modes assumed. In a later work, Dokuchaev[2] used a one-mode approximation to study the motion of a spinning satellite with five elastic rods. Recently, Brown and Schlack[3] used assumed modes to study the stability of a spinning satellite containing an elastic membrane. Other authors including Robe and Kane,[4] Likins and Wirsching,[5] and Nelson and Meirovitch[6] have studied the dynamics of satellites with elastic parts using either assumed modes to discretize the system or considered a discrete model from the beginning. Two major points must be made concerning the assumed mode technique. The first is that there is some uncertainty as to the effect of the series truncation on the stability criteria derived. The second is that, for a spinning system such as a satellite with elastic parts, the so-called "normal modes" which are assumed to represent the elastic displacements are usually those of the nonrotating system. These modes are not orthogonal modes for the spinning system and hence do not uncouple the system. The implication is that the assumed modes procedure represents merely a discretization scheme and not a modal analysis. Quite recently, Meirovitch[7,8] has developed a genuine modal analysis leading to a set of independent system of equations in terms of the system principal coordinates. This modal analysis permits the derivation of a closed-form solution to the response problem. In the case in which the problem is concerned with the system stability alone, however, it is not actually necessary to work with principal coordinates, although the stability analysis becomes extremely simple when principal coordinates are used. We shall clarify this statement later in this paper.

In a first attempt to apply Liapunov's direct method to hybrid systems from the area of satellite dynamics, Meirovitch[9,10] studied the stability of a spinning rigid body with elastic appendages. Several new concepts were introduced in Ref. 9 such as the use of the bounding properties of Rayleigh's quotient to eliminate the dependence of the testing functional on the spatial derivatives, as well as the concept of a testing density function. Reference 11 extends the development of Refs. 9 and 10 to hybrid systems with multielastic domains, such as a spinning rigid body with n rigidly attached flexible appendages simulating an orbiting satellite. The mathematical formulation consists of a hybrid set of $6(n+1)$ Hamiltonian equations of motion, six of which are ordinary differential equations for the rotational motion and $6n$ partial differential equations for the elastic motion. The stability analysis follows the pattern of Ref. 9, in which it is shown that under certain circumstances the system Hamiltonian H is a suitable Liapunov functional. The problem is shown to simplify considerably in the case in which it is possible to define density functions. However, it should be pointed out that this is not always possible.

Another approach to the stability of hybrid systems, based on the formulation of Ref. 11, involves the definition of new generalized coordinates representing certain integrals appearing in the testing functional as well as the use of Schwarz's inequality for functions. The method, referred to as "the method of integral coordinates" was introduced by the authors of this paper in Refs. 12 and 13. The method was compared to the assumed modes method and to the method of testing density functions in Ref. 14. It should be pointed out that Refs. 12 through 14 were concerned with systems possessing only one-dimensional elastic domains. A somewhat similar approach, applicable to certain problems involving the stability of bodies with fluid-filled cavities was presented in Ref. 15.

This paper uses the formulation of Ref. 11 and presents an extension of the method of integral coordinates developed in Refs. 12 and 13 to force-free systems with two- or three-dimensional elastic domains. In particular, the problem of the attitude stability of a spinning satellite consisting of a main rigid body containing an elastic membrane is solved. This is the same model as that of Ref. 3. Using the method of integral

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coordinates closed-form stability criteria are derived. These criteria lend themselves to ready physical interpretation. In addition, it is shown that the effect of the rotational motion on the natural frequencies of the elastic membrane cannot be ignored as assumed in Ref. 3, as it has a significant effect on the stability of the system.

II. Problem Formulation

The mathematical formulation, including the stability analysis for a hybrid dynamical system with multielastic domains, is essentially that given in Ref. 11. It will not be repeated here, but only the main features summarized.

Let us consider a body consisting of a rigid part occupying the domain D_0 and n flexible parts, rigidly attached to D_0 and occupying the domains D_i $(i = 1, 2, \ldots, n)$ when in undeformed state. The domains D_i have common boundaries only with D_0 (see Fig. 1). Denoting by D the domain of extension of the entire body, we have

$$D = \sum_{i=0}^{n} D_i$$

Let m_i $(i = 0, 1, \ldots, n)$ be the masses associated with the domains D_i, so that the total mass of the system is

$$m = \sum_{i=0}^{n} m_i$$

We shall be concerned with the motion of m relative to the inertial space XYZ. Let 0 be the origin of a system xyz, such that 0 is the mass center of m and axes xyz are the principal axes of m when in undeformed state. Note that system xyz is embedded in D_0 but may not be a set of principal axes for D_0. Let us also define sets of axes $x_i y_i z_i$ fixed relative to D_i $(i = 1, 2, \ldots, n)$ and with directions such that the elastic deformations are parallel to these axes for each domain. When the body deforms the mass center of m no longer coincides with 0 in general. We shall denote the mass center of the body in deformed state by c and sets of axes parallel to xyz and $x_i y_i z_i$ but with the origin at c instead of 0 by $\xi\eta\zeta$ and $\xi_i \eta_i \zeta_i$, respectively.

Next denote the radius vector from 0 to a given point in D_i $(i = 0, 1, \ldots, n)$ by $\mathbf{r}_i$, where the point in question coincides with the position of a mass element dm_i when the body is in undeformed state. Denoting by $\mathbf{i}_i, \mathbf{j}_i, \mathbf{k}_i$ the unit vectors along axes x_i, y_i, z_i, respectively, the position vector $\mathbf{r}_i$ can be written as

$$\mathbf{r}_i = x_i \mathbf{i}_i + y_i \mathbf{j}_i + z_i \mathbf{k}_i \qquad (i = 0, 1, \ldots, n)$$

The elastic displacement vector $\mathbf{u}_i$ of element dm_i depends on that position, as well as time, so that

$$\mathbf{u}_i = u_i(x_i, y_i, z_i, t)\mathbf{i}_i + v_i(x_i, y_i, z_i, t)\mathbf{j}_i + w_i(x_i, y_i, z_i, t)\mathbf{k}_i \qquad (i = 1, 2, \ldots, n)$$

The displacements u_i, v_i, w_i, measured along axes x_i, y_i, z_i, respectively, *are regarded as infinitesimally small.* Denoting by $\mathbf{r}_c$ the vector from 0 to c, we conclude that $\mathbf{u}_{ci} = \mathbf{u}_i - \mathbf{r}_c$ represents the displacement vector of the element of mass dm_i relative to c.

Omitting several intermediate steps, which are given in Ref. 11, we can write the kinetic energy in the matrix form

$$T = \tfrac{1}{2} m \{\dot{R}_c\}^T \{\dot{R}_c\} + \tfrac{1}{2} \sum_{i=0}^{n} \{\omega\}^T [l_i]^T [J_i][l_i]\{\omega\} + \{\omega\}^T \sum_{i=1}^{n} \int_{m_i} [r_i^{(0)} + u_{ci}^{(0)}][l_i]^T \{\dot{u}'_{ci}\}\, dm_i + \tfrac{1}{2} \sum_{i=1}^{n} \int_{m_i} \{\dot{u}'_{ci}\}^T \{\dot{u}'_{ci}\}\, dm_i \tag{1}$$

where $\{\dot{R}_c\}$ is a column matrix whose elements represent the velocity components of the mass center c, $\{\omega\}$ the angular velocity matrix of axes $\xi\eta\zeta$ relative to an inertial space, and $[l_i]$ the square matrix of direction cosines between $\xi_i \eta_i \zeta_i$ and $\xi\eta\zeta$. Moreover, $[J_i]$ is the inertia dyadic of m_i in deformed state expressed in terms of components about $\xi_i \eta_i \zeta_i$, $[r_i^{(0)} + u_{ci}^{(0)}]$ a skew-symmetric matrix whose elements are given by the relation

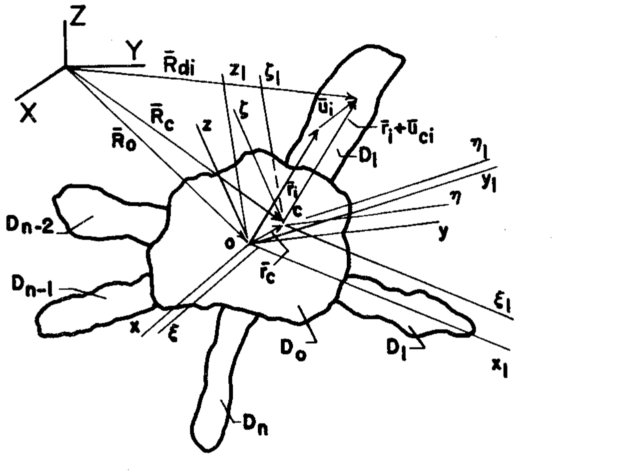

Fig. 1 The rigid body with flexible appendages.

$$r_{inm}^{(0)} + u_{cinm}^{(0)} = \sum_{l=1}^{3} \varepsilon_{mnl}(r_{il}^{(0)} + u_{cil}^{(0)}),$$

in which the superscript 0 indicates the base, ε_{mnl} is the epsilon symbol (see Ref. 16, p. 109), and $\{\dot{u}'_{ci}\}$ the matrix of elastic velocities of dm_i relative to c obtained by letting $\{\omega\} = \{0\}$. Assuming that changes in the gravitational potential energy are ignorable and that the components of displacement corresponding to one elastic domain are not affected by those corresponding to other elastic domains, the elastic potential can be written in the general functional form

$$V_{EL} = \sum_{i=1}^{n} V_{ELi} = \sum_{i=1}^{n} \int_{Di} \hat{V}_{EL}^{(i)} \left(\frac{\partial u_{ci}}{\partial x_i}, \frac{\partial u_{ci}}{\partial y_i}, \frac{\partial u_{ci}}{\partial z_i}, \frac{\partial v_{ci}}{\partial x_i}, \ldots, \frac{\partial^2 u_{ci}}{\partial x_i^2}, \frac{\partial^2 u_{ci}}{\partial x_i \partial y_i}, \ldots, \frac{\partial^2 w_{ci}}{\partial z_i^2} \right) \mathrm{d}D_i \tag{2}$$

where $\hat{V}_{EL}^{(i)}$ is the elastic potential energy density at a given point of domain D_i. Hence, the system has the general form

$$L = T - V_{EL} = L^{(0)}(\theta_1, \theta_2, \ldots, \dot{\theta}_3) + \sum_{i=1}^{n} \int_{D_i} \hat{L}^{(i)} \left(\theta_1, \theta_2, \ldots, \dot{\theta}_3, u_{ci}, v_{ci}, \ldots, \dot{w}_{ci}, \frac{\partial u_{ci}}{\partial x_i}, \frac{\partial u_{ci}}{\partial y_i}, \ldots, \frac{\partial w_{ci}}{\partial z_i}, \frac{\partial^2 u_{ci}}{\partial x_i^2}, \frac{\partial^2 u_{ci}}{\partial x_i \partial y_i}, \ldots, \frac{\partial^2 w_{ci}}{\partial z_i^2} \right) \mathrm{d}D_i \tag{3}$$

in which $L^{(0)}$ is the portion of the Lagrangian associated with the rigid part and $\hat{L}^{(i)}$ is a Lagrangian density.

As pointed out earlier, for the class of systems under consideration, a stability analysis based on the Liapunov direct method uses the Hamiltonian as a Liapunov functional. For the case of the hybrid system at hand, the Hamiltonian is defined by

$$H = \sum_{i=1}^{3} \frac{\partial L}{\partial \dot{\theta}_j} \dot{\theta}_j + \sum_{i=1}^{n} \int_{D_i} \left(\frac{\partial \hat{L}^{(i)}}{\partial \dot{u}_{ci}} \dot{u}_{ci} + \frac{\partial \hat{L}^{(i)}}{\partial \dot{v}_{ci}} \dot{v}_{ci} + \frac{\partial \hat{L}^{(i)}}{\partial \dot{w}_{ci}} \dot{w}_{ci} \right) \mathrm{d}D_i - L = H^{(0)}(\theta_1, \theta_2, \ldots, p_{\theta 3}) + \sum_{i=1}^{n} \int_{D_i} \hat{H}^{(i)} \left(\theta_1, \theta_2, \ldots, p_{\theta 3}, u_{ci}, v_{ci}, \ldots, \hat{p}_{wci}, \frac{\partial u_{ci}}{\partial x_i}, \frac{\partial u_{ci}}{\partial y_i}, \ldots, \frac{\partial w_{ci}}{\partial z_i}, \frac{\partial^2 u_{ci}}{\partial x_i^2}, \frac{\partial^2 u_{ci}}{\partial x_i \partial y_i}, \ldots, \frac{\partial^2 w_{ci}}{\partial z_i^2} \right) \mathrm{d}D_i \tag{4}$$

in which the meaning of $H^{(0)}$ and $\hat{H}^{(i)}$ is self-evident.

III. Stability Analysis

The stability analysis to be used here was developed in Ref. 11. We present here only the main features. It is well known that under special circumstances the Hamiltonian H is a suitable Liapunov functional for the class of problems under consideration.[9] For natural systems, the Hamiltonian is equal to the system total energy. In our case

$$H = T + V_{EL} \tag{5}$$

and, moreover, $\dot{H}$ is a negative semidefinite function in the generalized velocities and the set of points at which $\dot{H}$ is zero contains no nontrivial positive half-trajectory. It follows that, if H is positive definite, then the trivial solution is asymptotically stable.

Because H contains integrals involving the elastic displacements and their spatial derivatives it is a functional and not a function, so that its sign properties cannot be determined readily. Reference 11 presents a stability theorem particularly suited to this type of problems. The stability theorem can be stated as follows: If a positive definite function κ bounding the Hamiltonian H from below can be found, then the trivial solution is asymptotically stable. Clearly, the object of the theorem is to replace the functional H whose sign properties are difficult to ascertain by a function κ whose sign properties are relatively easy to determine, where $\kappa \leqq H$.

Using the bounding properties of Rayleigh's quotient, we can construct a functional V_{EL}^* such that

$$V_{EL} \geqq V_{EL}^* \tag{6}$$

where V_{EL}^* depends on the elastic displacements alone and not on their spatial derivatives. Hence, introducing the functional

$$\kappa^* = T + V_{EL}^* \tag{7}$$

we conclude that the system is stable if κ^* is positive definite. We note that κ^* is still a functional, so that the problem is still a hybrid one. Later in this paper we shall discretize the problem by constructing a function κ to be used with the stability theorem previously stated where κ is such that $\kappa \leqq \kappa^*$.

IV. Force-Free Systems

We shall be concerned with the attitude stability of a spinning spacecraft free of external forces, so that the torque about the mass center c is zero. It follows that the angular momentum vector about c is constant, so that it represents a motion integral. This integral can be used to reduce the order of the system.

From Ref. 11, we conclude that, if the momentum integral is taken into account, the functional κ^* can be written in the form

$$\kappa^* = T_2 + T_0 + V_{EL} \tag{8}$$

where T_2 is the portion of the kinetic energy quadratic in the generalized velocities and T_0 is free of generalized velocities. Assuming that the spacecraft spins initially about the principal axis z with the constant angular velocity Ω, where z is aligned with the inertial axis Z, we can write[11]

$$T_0 = \tfrac{1}{2} C^2 \Omega^2 \{l_Z\}^T [K] \{l_Z\} \tag{9}$$

where $[K]$ is the inverse of the inertia matrix $[J]$ of the entire body in deformed state and $\{l_Z\}$ is the column matrix of direction cosines $l_{\xi Z}, l_{\eta Z}, l_{\zeta Z}$ between the inertial axis Z and axes ξ, η, ζ, respectively. But, by definition, the quadratic part of the kinetic energy, T_2, is positive definite in the generalized velocities. It follows that κ^* is positive definite if $\kappa_0^* = T_0 + V_{EL}^*$ is positive definite in the generalized coordinates.

As the interest lies in the stability of the perturbed motion in the neighborhood of the equilibrium, we shall retain only those quantities in $[K]$ and $\{l_Z\}$ that do not yield terms of order higher than two in the generalized coordinates. With this in mind, we can write the matrix $[J]$ in the form

$$[J] = [J]_0 + [J]_1 + [J]_2 = [J]_0([1] + [J]_0^{-1}[J]_1 + [J]_0^{-1}[J]_2) \tag{10a}$$

where $[1]$ is the unit matrix of order three, $[J]_0$ is the inertia matrix of the body in the equilibrium state,

$$[J]_1 = \sum_{i=1}^{n} [l_i]^T [J_i]_1 [l_i] \tag{10b}$$

is the change in the inertia matrix of first order in the elastic displacements, and

$$[J]_2 = \sum_{i=1}^{n} [l_i]^T [J_i]_2 [l_i] \tag{10c}$$

is the change in the inertia matrix of second order, in which $[l_i]$ is the matrix of direction cosines between the local coordinates associated with the ith member and the coordinates $\xi\eta\zeta$. Note that $[J]_0$ can contain static elastic displacements caused by centrifugal forces resulting from steady spin, in which case the elastic terms in $[J]_1$ and $[J]_2$ represent the oscillations about the deformed equilibrium configuration. In view of this, if we regard $[J]_1$ and $[J]_2$ as small compared to $[J]_0$, then we can write the following approximate expression for the inverse of $[J]$

$$[K] = [J]^{-1} = ([1] + [J]_0^{-1}[J]_1 + [J]_0^{-1}[J]_2)^{-1}[J]_0^{-1} \cong [K]_0 + [K]_1 + [K]_2 \tag{11a}$$

where

$$\begin{aligned} [K]_0 &= [J]_0^{-1} \\ [K]_1 &= -[J]_0^{-1}[J]_1[J]_0^{-1} \\ [K]_2 &= -[J]_0^{-1}[J]_2[J]_0^{-1} + [J]_0^{-1}[J]_1[J]_0^{-1}[J]_1[J]_0^{-1} \end{aligned} \tag{11b}$$

Moreover, using results of Ref. 10, the matrix of direction cosines between axes $\xi\eta\zeta$ and Z can be approximated by

$$\{l_Z\} = \begin{Bmatrix} -\cos\theta_1 \sin\theta_2 \\ \sin\theta_1 \\ \cos\theta_1 \cos\theta_2 \end{Bmatrix} \cong \{l_Z\}_0 + \{l_Z\}_1 + \{l_Z\}_2 \tag{12a}$$

where

$$\{l_Z\}_o = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}, \{l_Z\}_1 = \begin{Bmatrix} -\theta_2 \\ \theta_1 \\ 0 \end{Bmatrix}, \{l_Z\}_2 = -\tfrac{1}{2} \begin{Bmatrix} 0 \\ 0 \\ \theta_1^{\,2} + \theta_1^{\,2} \end{Bmatrix} \tag{12b}$$

where θ_3, θ_1, and θ_2 represent rotations about ζ, ξ, and η, in that order. Note that θ_3 is an ignorable coordinate, a fact taken into account automatically in Eq. (9). Finally, considering Eqs. (11a) and (12a) and ignoring equilibrium terms, we can write an expression for κ_0^* valid in the neighborhood of the equilibrium in the form

$$\kappa_0^*|_E = \tfrac{1}{2} C^2 \Omega^2 (\{l_Z\}_1^T [K]_0 \{l_Z\}_1 + 2\{l_Z\}_2^T [K]_0 \{l_Z\}_0 + 2\{l_Z\}_1^T [K]_1 \{l_Z\}_0 + \{l_Z\}_0^T [K]_2 \{l_Z\}_0) + V_{EL}^* \tag{13}$$

The problem of investigating stability reduces to that of testing $\kappa_0^*|_E$, Eq. (13), for sign definiteness. However, it should be pointed out that $\kappa_0^*|_E$ is still a functional, which precludes its testing for sign definiteness by standard means. This problem can be circumvented by defining new coordinates representing certain integrals appearing in $\kappa_0^*|_E$ and using Schwarz's inequality for functions to discretize $\kappa_0^*|_E$. Then we can test the sign positiveness of the Hessian matrix $[H]_E$ associated with the resulting quadratic form by means of Sylvester's criterion (see Ref. 16, Sec. 6.4). Note that if modal analysis were to be used, i.e., if Eq. (13) were to be expressed in terms of the system principal coordinates, the Hessian matrix would become diagonal. Then it would be possible to determine the sign properties of the Hessian matrix by merely examining the sign of the diagonal elements of the matrix. Modal analysis is not really necessary for stability analysis, as the sign properties of the Hessian matrix can be investigated even if the matrix is not diagonal. This small digression was made, however, simply to point out the difference between the assumed modes method, which was frequently used as a discretization scheme alone but referred to as modal analysis, and the genuine modal analysis of Refs. 7 and 8. At this point it appears desirable to abandon generalities and consider a specific example.

V. Stability of the High-Spin Motion of a Satellite Containing an Elastic Membrane

Let us concern ourselves with the stability of a spinning satellite consisting of a main rigid body containing a square elastic membrane (see Fig. 2). The membrane is oriented such that its centroidal axes are parallel to the principal axes of the spacecraft in the equilibrium state, which in this particular case coincides with the undeformed state. Note that this is the same example as that treated in Ref. 3.

We shall assume that the motion of the mass center relative to the main body is zero, an assumption which will be shown later to be valid, so that the subscript c can be dropped from the elastic displacements. Because the membrane centroidal axes are parallel to the principal axes of the main body, we have $[l_1] = [1]$. Denoting by A, B, and C the principal moments of inertia of the entire spacecraft (including the membrane) when in the equilibrium state, we can write simply

$$[J]_0 = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} \tag{14a}$$

Moreover, because there is only one elastic domain involved, Eqs. (10b) and (10c) yield

$$[J]_1 = [l_1][J_1]_1[l_1] = \begin{bmatrix} 0 & 0 & -\int_D \rho xw\,dD \\ 0 & 0 & -\int_D \rho yw\,dD \\ -\int_D \rho xw\,dD & -\int_D \rho yw\,dD & 0 \end{bmatrix} \tag{14b}$$

$$[J]_2 = [l_1][J_1]_2[l_1] = \begin{bmatrix} \int_D \rho w^2\,dD & 0 & 0 \\ 0 & \int_D \rho w^2\,dD & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{14c}$$

where ρ is the mass per unit area of membrane and w represents the transverse displacement of the membrane of any point x, y. Inserting Eqs. (14) into Eq. (11), we have

$$[K]_0 = \begin{bmatrix} \frac{1}{A} & 0 & 0 \\ 0 & \frac{1}{B} & 0 \\ 0 & 0 & \frac{1}{C} \end{bmatrix} \tag{15a}$$

$$[K]_1 = \begin{bmatrix} 0 & 0 & \frac{1}{AC}\int_D \rho xw\,dD \\ 0 & 0 & \frac{1}{BC}\int_D \rho yw\,dD \\ \frac{1}{AC}\int_D \rho xw\,dD & \frac{1}{BC}\int_D \rho yw\,dD & 0 \end{bmatrix} \tag{15b}$$

$$[K]_2 = \begin{bmatrix} \frac{1}{A^2}\int_D \rho w^2\,dD & 0 & 0 \\ 0 & \frac{1}{B^2}\int_D \rho w^2\,dD & 0 \\ 0 & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} \frac{1}{A^2C}\left(\int_D \rho xw\,dD\right)^2 & \frac{1}{ABC}\left(\int_D \rho xw\,dD\right)\left(\int_D \rho yw\,dD\right) & 0 \\ \frac{1}{ABC}\left(\int_D \rho xw\,dD\right)\left(\int_D \rho yw\,dD\right) & \frac{1}{B^2C}\left(\int_D \rho yw\,dD\right)^2 & 0 \\ 0 & 0 & \frac{1}{AC^2}\left(\int_D \rho xw\,dD\right)^2 + \frac{1}{BC^2}\left(\int_D \rho yw\,dD\right)^2 \end{bmatrix} \tag{15c}$$

Finally, introducing Eqs. (12b) and (15) into Eq. (13), we obtain

$$\kappa_0^*|_E = \frac{1}{2}\Omega^2\left[\frac{C}{B}(C-B)\theta_1^2 + \frac{C}{A}(C-A)\theta_2^2 + 2\frac{C}{B}\theta_1\int_D \rho yw\,dD - 2\frac{C}{A}\theta_2\int_D \rho xw\,dD + \frac{1}{A}\left(\int_D \rho xw\,dD\right)^2 + \frac{1}{B}\left(\int \rho yw\,dD\right)^2\right] + V_{EL}^* \tag{16}$$

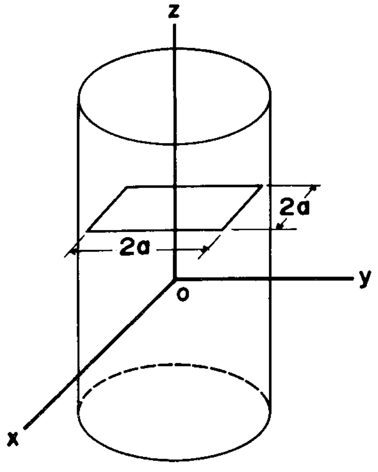

Fig. 2 Satellite containing an elastic membrane.

Next we must consider the term V_{EL}^*. To this end, we write first the expression for the elastic potential energy

$$V_{EL} = \frac{1}{2}\int_D \rho\left[T(x)\left(\frac{\partial w}{\partial x}\right)^2 + T(y)\left(\frac{\partial w}{\partial y}\right)^2\right]dD \tag{17}$$

where the tensions in the x and y directions are

$$T(x) = T + \tfrac{1}{2}\rho\Omega^2 a^2[1-(x/a)^2]$$
$$T(y) = T + \tfrac{1}{2}\rho\Omega^2 a^2[1-(y/a)^2] \tag{18}$$

in which T represents a constant initial tension and $\frac{1}{2}\rho\Omega^2a^2[1-(x/a)^2]$ and $\frac{1}{2}\rho\Omega^2a^2[1-(y/a)^2]$ represent increases in tension caused by the centrifugal forces.

The transverse displacement of the membrane is separable in space and time

$$w(x,y,t) = W(x,y)F(t) \tag{19}$$

where $W(x,y)$ satisfies the eigenvalue problem defined by the differential equation

$$\frac{\partial}{\partial x}\left[T(x)\frac{\partial W}{\partial x}\right] + \frac{\partial}{\partial y}\left[T(y)\frac{\partial W}{\partial y}\right] = \Lambda^2\rho W,$$
$$-a < x < a, \qquad -a < y < a \tag{20a}$$

and the boundary conditions

$$W = 0 \quad \text{at} \quad x = \pm a$$
$$W = 0 \quad \text{at} \quad y = \pm a \tag{20b}$$

Assuming that the mass distribution is uniform, $\rho = \text{const}$, the solution of the eigenvalue problem can be shown to consist of eigenfunctions that are either even or odd functions of x and y, respectively (see Ref. 17, Sec. 6-3). Hence, we can write the displacement w in the form

$$w(x,y,t) = w_{ee} + w_{oe} + w_{eo} + w_{oo} \tag{21}$$

where the subscripts e and o denote even and odd, respectively, and the first subscript is associated with x and the second with y.

Considering Eq. (16), we note that there is no coupling between the attitude variables θ_1 and θ_2 and the w_{ee} and w_{oo} terms. Furthermore, we note that any coupling between θ_1 and w_{oe} or between θ_2 and w_{eo} is also absent. In addition, because of orthogonality of the eigenfunctions, no coupling is introduced between the terms w_{oo}, w_{ee}, w_{oe} or w_{eo} by the elastic potential energy.

It can be verified that the lowest frequency of the membrane is associated with a mode shape which is of the w_{ee} type. Due to the lack of coupling in the testing functional $\kappa_0{}^*|_E$ and because the lowest natural frequency of the membrane is associated with motion which is uncoupled from the rigid body rotational motion, the form of the function $V_{EL}{}^*$ can be simplified by considering only motion of the elastic membrane that is coupled with the rigid body motion, namely, w_{oe} and w_{eo}. Furthermore, we note that w_{oe} and w_{eo} are orthogonal to one another as well as to w_{ee} and w_{oo}. In view of this, we can write

$$V_{EL} \geqq V_{EL}{}^* = \frac{1}{2}\Lambda_{12}{}^2 \int_D \rho w_{oe}{}^2\, dD + \frac{1}{2}\Lambda_{21}{}^2 \int_D \rho w_{eo}{}^2\, dD \quad (22)$$

where $\Lambda_{12}(\Lambda_{21})$ is the natural frequency of the first mode antisymmetric (symmetric) in x and symmetric (antisymmetric) in y. We note that for a square membrane with symmetrically distributed parameters $\Lambda_{12} = \Lambda_{21}$. Substituting $V_{EL}{}^*$ from Eq. (22) into Eq. (16), we obtain

$$\kappa_o{}^*|_E = \frac{1}{2}\Omega^2 \Bigg[\frac{C}{B}(C-B)\theta_1{}^2 + \frac{C}{A}(C-A)\theta_2{}^2 - 2\frac{C}{A}\theta_2 \int_D \rho y w_{oe}\, dD + 2\frac{C}{B}\theta_1 \int_D \rho y w_{eo}\, dD + \frac{1}{A}\left(\int_D \rho x w_{oe}\, dD\right)^2 + \frac{1}{B}\left(\int_D \rho y w_{eo}\, dD\right)^2 \Bigg] + \frac{1}{2}\Lambda_{12}{}^2 \int_D \rho w_{oe}{}^2\, dD + \frac{1}{2}\Lambda_{21}{}^2 \int_D \rho w_{eo}{}^2\, dD \quad (23)$$

We note that Eq. (23) is both a function and a functional and hence its definiteness is not readily found by standard means. In the next section we propose to discretize $\kappa_0{}^*|_E$ by the method of integral coordinates.

VI. Method of Integral Coordinates

Let us define new generalized coordinates involving integrals appearing in $\kappa_0{}^*|_E$ as follows:

$$\bar{w}_x(t) = \int_D \rho x w_{oe}(x, y, t)\, dD \quad (24a)$$

$$\bar{w}_y(t) = \int_D \rho y w_{eo}(x, y, t)\, dD \quad (24b)$$

By using Schwarz's inequality for functions, in conjunction with these integral coordinates, we can convert the testing functional $\kappa_0{}^*|_E$ into the testing function $\kappa_0|_E$ whose positive definiteness ensures stability. Although this is essentially a discretization scheme, it does not involve any truncation and the results obtained by this method can be accepted with confidence.

Recalling Schwarz's inequality for functions, and concentrating on the first integral coordinate in Eqs. (24), it is not difficult to show that

$$\left(\int_D \rho x w_{oe}\, dD\right)^2 \leqq \int_D \rho x^2\, dD \int_D \rho w_{oe}{}^2\, dD \quad (25a)$$

$$\left(\int_D \rho y w_{eo}\, dD\right)^2 \leqq \int_D \rho y^2\, dD \int_D \rho w_{eo}{}^2\, dD \quad (25b)$$

Recalling the definitions of $\bar{w}_x$ and $\bar{w}_y$, Eqs. (24), and solving for $\int_D \rho w_{oe}{}^2\, dD$ and $\int_D \rho w_{eo}{}^2\, dD$ we obtain

$$\int_D \rho w_{oe}{}^2(x, y, t) \geqq \bar{w}_x{}^2(t)/I_x \quad (26a)$$

$$\int_D \rho w_{eo}{}^2(x, y, t) \geqq \bar{w}_y{}^2(t)/I_y \quad (26b)$$

where $I_x = \int_D \rho x^2\, dD$ and $I_y = \int_D \rho y^2\, dD$. For a square membrane of uniform mass density $I_x = I_y = I_t$.

Inserting inequalities (25) into the expression for $\kappa_0{}^*|_E$, Eq. (23), and considering the definitions (24) for the integral coordinates, we obtain the testing function

$$\kappa_0|_E = \frac{1}{2}\Omega^2 \Bigg[\frac{C}{B}(C-B)\theta_1{}^2 + \frac{C}{A}(C-A)\theta_2{}^2 - 2\frac{C}{A}\theta_2 \bar{w}_x + 2\frac{C}{B}\theta_1 \bar{w}_y + \left(\frac{1}{A} + \frac{\Lambda_{12}{}^2}{\Omega^2}\frac{1}{I_t}\right)\bar{w}_x{}^2 + \left(\frac{1}{B} + \frac{\Lambda_{21}{}^2}{\Omega^2}\frac{1}{I_t}\right)\bar{w}_y{}^2 \Bigg] \quad (27)$$

where $\kappa_0|_E \leqq \kappa_0{}^*|_E$. Hence if $\kappa_0|_E$ is positive definite the trivial solution is asymptotically stable.

Before continuing with the stability analysis, we note that the method of integral coordinates as applied to multi-dimensional elastic domains requires that two or three new coordinates be defined for each elastic displacement where the number of new coordinates is dictated by the number of spatial variables. Specifically, for the case at hand, there are two new coordinates, $\bar{w}_x$ and $\bar{w}_y$, associated with the elastic displacement w corresponding to the spatial variables x and y. In general, this requires that the elastic potential energy be broken down such that one part of the potential energy is associated with each integral coordinate. For the problem at hand this separation occurs quite naturally, but such a separation is possible provided the elastic member represents a self-adjoint system.[17]

Returning to the stability analysis we note that $\kappa_0|_E$ is the sum of two independent quadratic forms, each of which must be positive definite. Denoting these forms by $\kappa_{01}|_E$ and $\kappa_{02}|_E$ and and their associated Hessian matrices by $[H_{01}]$ and $[H_{02}]$, respectively, we obtain

$$[H_{01}] = \frac{1}{2}\begin{bmatrix} \frac{C}{A}(C-A) & -\frac{C}{A} \\ -\frac{C}{A} & \left(\frac{1}{A} + \frac{\Lambda_{12}{}^2}{\Omega^2}\frac{1}{I_t}\right) \end{bmatrix} \quad (28a)$$

$$[H_{02}] = \frac{1}{2}\begin{bmatrix} \frac{C}{B}(C-B) & \frac{C}{B} \\ \frac{C}{B} & \left(\frac{1}{B} + \frac{\Lambda_{21}{}^2}{\Omega^2}\frac{1}{I_t}\right) \end{bmatrix} \quad (28b)$$

The fact that $\kappa_0|_E$ can be expressed as the sum of two independent quadratic forms can be traced to the fact that $\kappa_0|_E$ does not contain cross products of θ_1 and θ_2 on the one hand and of $\bar{w}_x$ and $\bar{w}_y$ on the other hand, which in turn is a direct result of the symmetry of the membrane. For more complicated systems the absence of cross products should not be expected. An application of Sylvester's criterion to matrices (28), yields the following stability criteria

$$C > A, \qquad \frac{\Omega}{\Lambda_{12}} < \left[\frac{C-A}{I_t}\right]^{1/2} \quad (29a)$$

$$C > B, \qquad \frac{\Omega}{\Lambda_{21}} < \left[\frac{C-B}{I_t}\right]^{1/2} \quad (29b)$$

We note that criteria (29b) can be obtained from criteria (29a) by replacing A by B and Λ_{12} by Λ_{21}. In the following, we shall consider only criteria (29a), bearing in mind that analogous results can be obtained from criteria (29b) if the replacements mentioned are made. For convenience we define the $R_I = I_t/A_0$, where $A_0 = A - I_t$ is the moment of inertia of main body alone about ξ. With these definitions, criteria (29a) become

$$C > A, \qquad \frac{\Omega}{\Lambda_{12}} < \left[\frac{(C_0/A_0) - 1 + R_I}{R_I}\right]^{1/2} \quad (30)$$

where $C_0 = C - 2I_t$ is the moment of inertia of the main body alone about ζ.

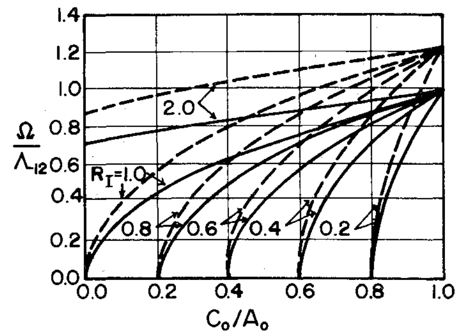

Fig. 3 Stability regions with R_I as a parameter.

Two immediate conclusions can be drawn from inequalities (30): 1) for spin stabilization the spinning motion should be imparted about the axis of maximum moment of inertia; and 2) spin stabilization is possible if the spin ratio Ω/Λ_{12} satisfies the second of inequalities (30) involving the system parameters R_I and C_0/A_0. The conclusions are in line with those obtained in Refs. 10 and 12.

VII. Numerical Results

The numerical results are exhibited in the form of stability diagrams for the second of inequalities (30). Figure 3 shows the spin ratio Ω/Λ_{12} required for stability as a function of C_0/A_0 with R_I as a parameter. The region below the appropriate curve is stable. For comparison purposes, the dashed lines on Fig. 3 represent results obtained by approximating the elastic displacement w with four assumed modes. The ratio Ω/Λ_{12} vs $\Omega/(\Lambda_{12})_{NR}$ where $(\Lambda_{12})_{NR}$ is the frequency of the nonrotating membrane obtained by setting $\Omega = 0$. It should be noted that Fig. 4 shows that for $\Omega/(\Lambda_{12})_{NR} < 2$ the value of the ratio Ω/Λ_{12} is always smaller than approximately 0.55. Consequently, the relatively large difference between the results obtained by the assumed modes method and those obtained by the integral coordinates method does not actually exist. Indeed, for the possible values of Ω/Λ_{12} given in Fig. 4, we conclude that Fig. 3 shows excellent agreement between the results obtained by the two methods.

VIII. Summary and Conclusions

The mathematical formulation associated with the problem of the stability of motion of a satellite containing two- or three-dimensional elastic domains has been completed. Whereas, the rotational motion of the body is described by generalized coordinates depending on time alone, the elastic displacements depend on both spatial position and time. In the absence of external forces, there exist motion integrals in the form of momentum integrals. These integrals can be regarded as constraint equations relating the system angular velocities.

The Liapunov direct method has been chosen for the stability analysis because it is likely to yield results which can be interpreted more readily than those obtained by a purely numerical integration of the equations of motion. Since the elastic vibration results in energy dissipation, the equilibrium is asymptotically stable if the Hamiltonian is positive definite; it is unstable if it can take on negative values in the neighborhood of equilibrium. Determining the sign definiteness of the Hamiltonian is complicated by the fact that it is both a function and a functional at the same time. The method of integral coordinates is particularly suited to treat this problem. The method yields closed-form stability criteria involving the system parameters. In the case of the system containing a membrane, these parameters consist of body and membrane moments of inertia, and the satellite spin velocity. The advantage of the method of integral coordinates is demonstrated by the relative ease with which closed-form stability criteria are developed. The analysis shows that for stability the spinning motion must be imparted about the axis of maximum moment of inertia and that the allowable spin ratio Ω/Λ_{12} depends on the system parameters.

For comparison purposes, the same problem has been solved by the assumed modes method, whereby the admissible functions used to discretize the system are the natural modes of the uniformly rotating membrane. This analysis shows that the natural frequencies of the spinning membrane is greatly affected by the centrifugal stiffening. Hence, the assumption of Ref. 3 that the edge tension is large in comparison to the centrifugal forces is invalid. Specifically, all points predicted to be unstable in the stability plot of Ref. 3 represent spin ratio values of $\Omega/(\Lambda_{12})_{NR} > 1$; from Fig. 4, however, we conclude that these spin ratios are in error by at least a factor of two.

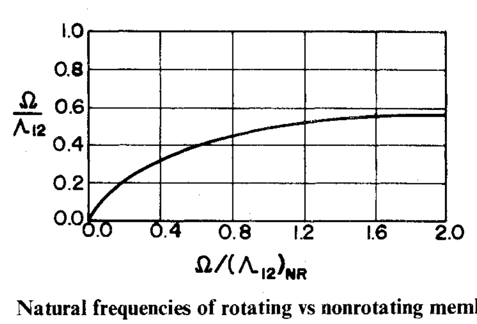

Fig. 4 Natural frequencies of rotating vs nonrotating membranes.

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